Lecture 0 - ODE 1-3 Review

Monday, January 26, 2026 6:25 AM

1.1 - A differential equation is an equation relating an unknown function $y(x)$ and its derivatives.

- Order refers to the highest derivative present.

- An n-th order ODE is linear if it can be written

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = g(x),$$

where coefficients $a_k(x)$ and forcing $g(x)$ depend only on x (not on y or derivatives.

- A solution of an n-th order ODE is a function ϕ that has at least n-th derivatives and

$$F(x, \phi(x), \phi'(x), \dots, \phi^{(n)}(x)) = 0$$

for all $x \in I$, where I is an interval such that n-th derivative of ϕ exists and is continuous.

1.2 - A general solution contains arbitrary constants arbitrary constants

- A particular solution is obtained after imposing conditions.

2.1 - A separable ODE has the form $\frac{dy}{dx} = g(x)h(y) \Rightarrow \frac{1}{h(y)} dy = g(x)\,dx$.

Solve by integrating both sides, then apply the initial conditions.

2.2 - A first-order linear ODE is

$$y' + p(x)y = q(x).$$

Define the integrating factor:

$$\mu(x) = e^{\int p(x)\,dx}$$

Then

$$\mu(x)\,y(x) = \int \mu(x)\,q(x)\,dx + C.$$

2.3 - A first-order ODE written as

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ on a region.

Then there exists a potential function $\Phi(x,y)$ such that

$$\Phi_x = M, \quad \Phi_y = N,$$

and solutions are given by $\Phi(x,y) = C$.

2.4 Solutions by substitutions;

2.4 Solutions by substitution; Bernoulli's method, homogeneous equations in y/x, etc.

3.1 - Consider a linear homogeneous nth order ODE:

$$a_n(x)y^{(n)} + \dots + a_0(x)y = 0$$

If $y_1, \dots, y_n$ are linearly independent solutions, then

$$y = c_1 y_1 + \dots + c_n y_n$$

is the general solution.

- For y_1, y_2, define

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2.$$

If $W(x_0) \neq 0$, then y_1, y_2 are linearly independent.

3.2 - If you know one nonzero solution y_1 of a second-order linear homogeneous ODE, seek another as $y_2 = v(x)y_1(x)$, leading to an equation for $v'(x)$.

3.3 - For $ay'' + by' + cy = 0$, try $y = e^{rx}$ to get the characteristic equation try $y = e^{rx}$ to get the characteristic equation

$$ar^2 + br + c = 0.$$

Cases: - distinct real r_1, r_2:

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

- repeated real r_1, r_2:

$$y = c_1 e^{rx} + x c_2 e^{rx}$$

- complex $r = \alpha + i\beta$:

$$y = e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right)$$

3.4 - For $ay'' + by' + cy = g(x)$, when $g(x)$ is a combination of polynomials, exponentials, sines/cosines, try a particular solution of y_p with the same form.

- $g(x) = P_m(x)$: try $y_p = Q_m(x)$.
- $g(x) = e^{kx} P_m(x)$: try $y_p = e^{kx} Q_m(x)$.
- $g(x) = \sin(\omega x)$ or $\cos(\omega x)$: try $y_p = A\cos(\omega x) + B\sin(\omega x)$.

- if overlap is present with the homogeneous solution, multiply by x until it no longer exists.